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GENERAL INVESTIGATION OF THE CONVERGENCE OF TRIGONOMETRIC SERIES INTO WHICH ARBITRARY FUNCTIONS ARE EXPANDED AND SOME NEW APPLICATIONS OF THE SAME.

BY

JULIUS FRIEDLÄNDER, Dr. Phil.

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BERLIN.
1853.

Only 25 copies printed.

TO Dr. B. APTHORP GOULD

THIS MEMOIR IS RESPECTFULLY DEDICATED

BY HIS FRIEND

THE AUTHOR.

The great theoretical interest which attaches to the expansion of arbitrary functions into trigonometric series, commonly called Fourier's series, together with the extensive practical applications of the same, have long engaged the attention of Mathematicians. The summation of these series was first succeeded in by the famous *Lejeune-Dirichlet* after fruitless attempts by the most celebrated Analysts, and we have to thank the astronomer ***Bessel*** for the enlarged application of these developments to the exact sciences. This branch of investigation is still unusually rich in the most remarkable laws and inferences. It might seem rash to venture upon a field where a mind like that of *Lejeune-Dirichlet* has been engaged, were it not well known that when a masterhand has reared the building the ornamenting it is easy.

The author finds his chief inducement to publish this treatise in the fact that his celebrated master has proved the convergence of the series for less cases than the following method provides for. He would fain hope, moreover, that several applications of the theory which are quite new, will be of interest to the scientific world.

It is of importance for the following investigations to consider the consequences which follow from one or more inequations as,

$$a < b < c.$$

We write $p < q$, i. e. p less than q, when the algebraic difference $p - q$ is negative.

I. From the inequation

$$a < b < c$$

follows for every value of d

$$a + d < b + d < c + d.$$

II. From the inequations

$$a < b < c$$
$$a_1 < b_1 < c_1$$

follows,

$$a + a_1 < b + b_1 < c + c_1$$
$$a - a_1 < b - b_1 < c - c_1.$$

III. Again from the inequation

$$a < b < c$$

follows $ka < kb < kc$ when $k > 0$

$kc < kb < ka$ when $k < 0.$

IV. The absolute values of three quantities connected together by the inequation $a < b < c$ are subject to a certain law: Let the absolute or numerical value of a real number a be represented by $M(a)$, i. e. modulus of a, whence

$$M(a) = M - (a)$$

we can then derive from the inequation

$$a < b < c,$$

the new inequation $M(b) < M(a) + M(c)$.

For, whatever be the values of a, b, c, in the inequation

$$a < b < c$$

we must have either b and c positive,

or a and b negative.

The three values of a, b, c arranged in the order of their magnitudes can be considered as the abscissas of three points in the axis of abscissas. Three such points can have no other situation with respect to the origin of coordinates, of which the abscissa is 0, than:

$$\left.\begin{array}{c} 0,\ a,\ b,\ c \\ a,\ 0,\ b,\ c \end{array}\right\} \text{ in both cases } b \text{ and } c \text{ are positive}$$

$$\left.\begin{array}{c} a,\ b,\ 0,\ c \\ a,\ b,\ c,\ 0 \end{array}\right\} \text{ in both cases } a \text{ and } b \text{ are negative.}$$

When b and c are positive, we have $M(b) < M(c)$, when a and b are negative, we have $M(b) < M(a)$, then certainly in both cases, $M(b) < M(a) + M(c)$.

V.

$$M(a+c) \leqq M(a) + M(c)$$
$$M(a-c) \leqq M(a) + M(c)$$

i. e. the modulus of the sum or difference of two quantities is equal to or less than the sum of the moduli of the two quantities. This proposition is easily derived from a consideration of the different signs by which a and b can be affected.

§ 2. Let $f(x)$ be a continual or discontinual*) function of x which has between the limits $x = a$ and $x = b$ for every x, one, and only one determinate value $f(x)$. Let us assume between the arguments $x = a$ and $x = b$ an arbitrary finite or infinite number of arguments arranged in the order of their magnitudes:

$$a_1,\ a_2,\ a_3,\ \ldots.,\ a_m \ldots\ldots b$$

at equal or unequal intervals.

We will now form the sum

$$f(a_1) - f(a_2) + f(a_3) - f(a_4) + \ldots\ldots$$

call it *alternating sum* aad designate it by

$$\sum_a^b \pm f(x).$$

The following general law exists with regard to this alternating sum $\sum_a^b \pm f(x)$, although the number of terms in it be unknown or infinite, and their values not fixed.

The sum $\sum_a^b \pm f(x) = f(a_1) - f(a_2) + f(a_3) - f(a_4) \ldots\ldots.$ *has a finite value; and we can in fact ascertain a limit, independent of the number of terms, which the alternating sum cannot exceed; provided $f(x)$ be such a function as we have described in the beginning of this paragraph, and provided the number of maxima and minima between the limits $x = a$ and $x = b$ be finite, — that is to say the number which indicates how often $f(x)$ changes from increasing to decreasing and vice versa, be finite.*

We call $f(x)$ increasing between $x = p$ and $x = q$ if for the arguments $p,\ p_1,\ p_2, \ldots.\ q$ which follow each other in such a way that $p < p_1 < p_2 \dots\dots < q$,

---

*) To define the meaning of the expression $F(x)$ is discontinual and has but one value for $x = \delta$ we have to calculate the values of

$F(x_1),\ F(x_2),\ F(x_3) \ldots\ldots F(\delta)$

where $x_1 < x_2 < x_3 \ldots < \delta$

and the values $F(x'),\ F(x''),\ F(x''') \ldots.\ F(\delta)$

where $x' > x'' > x''' > \ldots\ldots. > \delta$,

If now $F(\delta)$ have two different values for the point $x = \delta$ [supposing $F(x)$ to represent the abscissas of a curve] according as we reach the point from the side of positive or negative abscissas, the function is called discontinual for $x = \delta$, and we can nevertheless say that $F(\delta)$ has but one value if we add from which side the point is reached. We represent this value by $F(+\delta)$ and $F(-\delta)$.

$$f(p) < f(p_1) < f(p_2) \dots\dots < f(q)$$

we call it decreasing, when under the same circumstances

$$f(p) > f(p_1) > f(p_2) \dots\dots > f(q).$$

To demonstrate this general law, let $f(x)$ be decreasing between $x = \mu$ and $x = \nu$. Assume that there are between $x = \mu$ and $x = \nu$ the two following series of arguments:

$$\text{I.}\quad \mu,\ m_1,\ m_2\ \dots\dots\ m_{2k},\ m_{2k+1},\ \nu$$

$$\text{II.}\quad \mu,\ m_1,\ m_2\ \dots\dots\ m_{2k},\ \nu$$

in the one case (I) an odd, in the other (II), an even number, it follows, since $m_1 < m_2 < m_3 \dots\dots$

$$(\text{A}) \quad \left\{ \begin{array}{ll} f(m_1) - f(m_2) & > 0 \\ f(m_3) - f(m_4) & > 0 \\ \vdots \qquad\quad \vdots & \\ f(m_{2k-}{}^{3}) - f(m_{2k-2}) & > 0 \\ f(m_{2k-1}) - f(m_{2k}) & > 0 \end{array} \right.$$

$$(\text{B}) \quad \left\{ \begin{array}{ll} -f(\mu) \ + f(m_1) & < 0 \\ -f(m_2) + f(m_3) & < 0 \\ \vdots \qquad\quad \vdots & \\ -f(m_{2k-2}) + f(m_{2k-1}) & < 0 \\ -f(m_{2k}) \quad + f(m_{2k+1}) & < 0 . \end{array} \right.$$

If we add the inequations of the two systems (A) and (B) it follows from (A), $f(m_1) - f(m_2) + f(m_3) - f(m_4) + \dots f(m_{2k-1}) - f(m_{2k}) > 0$

$$f(m_1) - f(m_2) + f(m_3) - f(m_4) + \dots f(m_{2k-1}) - f(m_{2k}) + f(m_{2k+1}) < f(\mu).$$

In case the number of arguments assumed between μ and ν be even, we have from (A)

$$0 < \sum_{\mu}^{\nu} \pm f(x)$$

and from (B) $\quad \displaystyle\sum_{\mu}^{\nu} \pm f(x) < f(\mu) - f(m_{2k+1})$

or since $\quad 0 < f(m_{2k+1}) - f(\nu)$

$$\sum_{\mu}^{\nu} \pm f(x) < f(\mu) - f(\nu)$$

and joined with the first inequation, we have for an even number of arguments between μ and ν:

$$\text{(C)} \qquad 0 < \sum_{\mu}^{\nu} \pm f(x) < f(\mu) - f(\nu).$$

In the second case where $+ f(m_{2k+1})$ still appears in the *alternating sum*, we derive from (B)

$$\sum_{\mu}^{\nu} \pm f(x) < f(\mu);$$

and from (A) $\quad \sum_{\mu}^{\nu} \pm f(x) > f(m_{2k+1}).$

or because $\quad 0 > - f(m_{2k+1}) + f(\nu),$

we have $\quad \sum_{\mu}^{\nu} \pm f(x) > f(\nu)$

and combined with the previous inequation it follows in case the number of interpolated arguments be uneven,

$$\text{(D)} \qquad f(\nu) < \sum_{\mu}^{\nu} \pm f(x) < f(\mu).$$

It is necessary to replace the two results (C) and (D) which we have obtained from these two cases, by a single one; opportunity is offered for this by § I. It follows from (C) according to § I, 4 that the numerical value of the alternating sum is less than the numerical value of $f(\mu) - f(\nu)$, likewise from (D) in the second case, that the numerical value of the alternating sum is less than the sum of the numerical values of $f(\mu)$ and $f(\nu)$. According to I, 5 the numerical value of $f(\mu) - f(\nu)$ i. e. $M[f(\mu) - f(\nu)] \leqq M[f(\mu)] + M[f(\nu)]$. It follows therefore, as well from (C) as from (D), that the numerical value of the alternating sum is less than, or equal to the sum of the numerical values of $f(\mu)$ and $f(\nu)$ i. e.

$$M[\sum_{\mu}^{\nu} \pm f(x)] \leqq M[f(\mu)] + M[f(\nu)].$$

We add here for the sake of completeness that in the alternating sum several succeeding terms might be equal, also that the extreme terms of the sum might be identical with $f(\mu)$ or $f(\nu)$, without affecting the truth of the law, inasmuch as the systems (A) and (B), will still hold good and therefore also their consequences.

Let $f(x)$ be increasing between $x = \pi$ and $x = \sigma$, than the function $- f(x)$ will decrease between $x = \pi$ and $x = \sigma$ and therefore the equation

$$M\left(\sum_{\pi}^{\sigma} \pm [-f(x)]\right) \leqq M[-f(\pi)] + M[-f(\sigma)]$$

be true for $-f(x)$

but $\sum_{\pi}^{\sigma} \pm [-f(x)] = -\sum_{\pi}^{\sigma} \pm f(x)$

and since $M(-a) = M(a)$
it follows for the alternating sum of $f(x)$ for x between π and σ if $f(x)$ increases between $x = \sigma$ and $x = \pi$ that

$$\left(\sum_{\pi}^{\sigma} \pm f(x)\right) \leqq M[f(\pi)] + M[f(\sigma)]$$

Therefore the alternating sum of a function between two values between which the function either decreases or increases, is finite and limited by the sum of two known values. We have made no hypothesis with respect to the properties of the function; it can therefore be continual or discontinual; it can be in fact infinitely often discontinual if the function either increase or decrease throughout.

Let all the arguments between a and b at which the functions passes from increasing to discreasing and vice versa be known.

Let these arguments be in their order

$$\mu_1,\ \mu_2,\ \mu_3 \ldots\ldots\ldots \mu_q$$

$$\text{Now } \sum_{a}^{b} \pm f(x) = \sum_{a}^{\mu_1} \pm f(x) + \sum_{\mu_1}^{\mu_2} \pm f(x) \ldots \sum_{\mu_q}^{b} \pm f(x)$$

moreover the modulus of $\sum_{a}^{b} \pm f(x) \leqq$ the sum of the moduli of the separate terms, i. e. of the separate alternating sums, each of which is known to us as being confined within certain limits; therefore

$$\begin{aligned} M\left[\sum_{a}^{b} \pm f(x)\right] \leqq\ & M[f(a)] + M[f(\mu_1)] \\ & + M[f(\mu_1)] + M[f(\mu_2)] \\ & + M[f(\mu_2)] \ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & \ldots\ldots\ldots\ldots + M[f(\mu_q)] \\ & + M[f(\mu_q)] + M[f(b)] \end{aligned}$$

If for no one of the above mentioned arguments μ_p the function be at the same time discontinual, we have always

$$M[f(\mu_p)] = M[f(\mu_p)]$$

(Compare what has been said in the end of pag. 3. of discontinuities.)

It is therefore proved that the numerical value of the alternating sum of a function between $x = a$ and $x = b$ is limited by the sum of the numerical values of $f(a)$ and $f(b)$ and twice the numerical values of its maxima and minima between $x = a$ and $x = b$. But in case a maximum or a minimum exist at the same point with a discontinuity, that is to say at a point where the function has two values according to the direction from which we reach this point, we must write for, "twice the numerical value" "the sum of the two different numerical values."

§ 3. According to the theorem immediately preceding we can easily ascertain when the alternating sum of a function $\varphi(x)$ between $x = a$ and $x = b$ is finite, and what its limits are. Let us represent for the sake of brevity those values of the function from which the function passes from decreasing and vice versa by the name of ***characteristic values***.

If the number of characteristic values of the function between $x = a$ and $x = b$ be finite, whatever the number be, then is the alternating sum certainly finite; but if their number be infinite, and if it can be proved that their numerical values form a converging series, so is it also proved that the alternating sum is finite and the limits can also be stated between which it is inclosed. If this cannot be proved, it remains uncertain whether the alternating sum is finite or not.

§ 4. We pass now to a new proposition. Let $f(x)$ be such a continual or discontinual function that for every value between $x = a$ and $x = b$ it has a certain finite value; moreover let the alternating sum of $f(x)$ between $x = a$ and $x = b$ be finite; i. e. the number of ***characteristic values*** of the function between $x = a$ and $x = b$ form a finite or their modulus form an infinite converging series, then is

$$\lim^{k=\infty} \int_a^b f(a) \sin kx dx = 0$$

If we make $kx = y$ and therefore $dx = \frac{1}{k} dy$ and change the limit accordingly, we have the equation

$$\lim^{k=\infty}\int_{a}^{b} f(x)\sin kx\,dx = \lim^{k=\infty}\frac{1}{k}\int_{ka}^{kb} f\left(\frac{y}{k}\right)\sin y\,dy.$$

Let us fix upon a certain value for k, ascertain for this the value of the integral and then we can obtain the value for increasing k. Decompose the integral into the sum of the following

$$\ldots. + \int_{(2n-1)\pi}^{2n\pi} f\left(\frac{y}{k}\right)\sin y\,dy + \int_{2n\pi}^{(2n+1)\pi} f\left(\frac{y}{k}\right)\sin y\,dy + \int_{(2n+1)\pi}^{(2n+2)\pi} f\left(\frac{y}{k}\right)\sin y\,dy + \ldots.$$

in which the limits are whole multiples of π. We discuss the value of each of these partial integrals in accordance with the well-known theorem: If $f(x)$ do not change its sign between $x = \alpha$ and $x = \beta$, and M and N are the greatest and least values which the function $\varphi(x)$ assumes in the interval $x = \alpha$ to $x = \beta$, then is

either $$N\int_{\alpha}^{\beta} f(x)\,dx < \int_{\alpha}^{\beta} \varphi(x) f(x)\,dx < M\int_{\alpha}^{\beta} f(x)\,dx$$

or $$M\int_{\alpha}^{\beta} f(x)\,dx < \int_{\alpha}^{\beta} \varphi(x) f(x)\,dx < N\int_{\alpha}^{\beta} f(x)\,dx$$

If we apply this theorem to each of the partial integrals for $f(x) = \sin\frac{\pi}{2}\,y$ does not change the sign between the limits we have, e. g.

$$\int_{(2\nu-1)\pi}^{2\nu\pi} f\left(\frac{y}{k}\right)dy$$

is contained between

$$n_{2\nu}\int_{(2\nu-1)\pi}^{2\nu\pi}\sin y\,dy \quad\text{and}\quad m_{2\nu}\int_{(2\nu-1)\pi}^{2\nu\pi}\sin y\,dy.$$

where $m_{2\nu}$ and $n_{2\nu}$ represent the greatest and least values which the function $f\left(\frac{y}{k}\right)$ assumes within the limits $y = 2n\pi$ and $y = (2n-1)\pi$.

But since $$\int_{(2\nu-1)\pi}^{2\nu}\sin y\,dy = -1$$

it follows that $\int\limits_{(2\nu-1)\pi}^{2\nu} f\left(\frac{y}{k}\right) \sin y \, dy$

is contained between $-m_{2\nu}$ and $-n_{2\nu}$.

It follows in like manner that

$$\int\limits_{2\nu}^{2\nu+1} f\left(\frac{y}{k}\right) \sin y \, dy$$

is contained between $+m_{2\nu+1}$ and $+n_{2\nu+1}$ inasmuch as $m_{2\nu+1}$ and $n_{2\nu+1}$ represent the greatest and least values which $f\left(\frac{y}{k}\right)$ assumes between $y = 2\nu$ and $y = 2\nu + 1$ etc.

The whole integral $\int\limits_{ka}^{kb} f\left(\frac{y}{k}\right) \sin y \, dy$

is therefore contained between two alternating sums, since $m_{2\nu}$ and $n_{2\nu}$, $m_{2\nu+1}$ and $n_{2\nu+1}$ etc. represent succeeding values of the function $f\left(\frac{y}{k}\right)$ between ka and kb, or of the function $f(y)$ between a and b. In accordance with the hypothesis which we have made with respect to the function $f(y)$ the alternating sum $\sum\limits_{a}^{b} \pm f(x)$ or $\sum\limits_{ka}^{kb} \pm f\left(\frac{y}{k}\right)$ is finite and contained between statable limits. Between the same finite limits is contained also the value of

$$\int\limits_{ka}^{kb} f\left(\frac{y}{k}\right) \sin y \, dy,$$

whatever be the value of k. Although k may exceed any value however great, the integral

$$\int\limits_{ka}^{kb} f\left(\frac{y}{k}\right) \sin y \, dy \qquad \text{remains finite and}$$

$$\lim^{k=\infty} \frac{1}{k} \int\limits_{ka}^{kb} f\left(\frac{y}{k}\right) \sin y \, dy = 0 \text{ . Q.E.D.}$$

In like manner it can be proved that

$$\lim^{k=\infty}\int_a^b \cos kx\,dx = 0,$$

$$\lim^{k=\infty}\int_a^b f(x)e^{kx\sqrt{-1}}\quad dx = 0,$$

and $$\lim^{k=\infty}\int_a^b f(x)e^{-kx\sqrt{-1}}\quad dx = 0.$$

§ 5. Let $\varphi(x)$ be such a function that it becomes infinite between $x = a$ and $x = b$. Let

$$\alpha_1\,,\ \alpha_2\,,\ \alpha_3\ \ldots\ldots\ \alpha_\nu\ \ldots\ \alpha_n$$

represent the n arguments between a and b for which

$$\varphi(x) = \infty.$$

But let $\frac{\varphi(x)}{x-\alpha_1}$ for $x_1 = \alpha_1$, $\frac{\varphi(x)}{x-\alpha_2}$ for $x = \alpha_2$ etc. have finite values; let us investigate what value

$$\lim^{k=\infty}\int_a^b \varphi(x)\sin dx\,kx$$

assumes. Having decomposed the integral $\int_a^b \varphi(x)\sin kdx$ in $2n+1$ integrals in such a way that the first, third, fifth etc. contain no root of the equation $\varphi(x) = \infty$ within its limits, and that the second, fourth etc. contain, each one; then is, if ε represent a proper positive quantity,

$$\int_a^b \varphi(x)\sin kx\,dx = \int_\alpha^{\alpha_1-\varepsilon}\varphi(x)\sin kx\,dx + \int_{\alpha_1-\varepsilon}^{\alpha_1+\varepsilon}\varphi(x)\sin kx dx + \int_{\alpha_1+\varepsilon}^{\alpha_2-1}\varphi(x)\sin kx\,dx + \ldots\ldots$$
$$+\int_{\alpha_n+\varepsilon}^{b}\varphi(x)\sin kx\,dx.$$

If k increase to infinity all those integrals which have within their limits no argument which is a root of the equation $\varphi(x) = \infty$, that is to say the first, third, fifth etc., will disappear if $\varphi(x)$ fulfil the condition that

$$\sum_{a}^{\alpha_1-\varepsilon} \pm \varphi(x) + \sum_{\alpha_1+\varepsilon}^{\alpha_2-\varepsilon} \pm \varphi(x) + \sum_{\alpha_2+\varepsilon}^{\alpha_3-\varepsilon} \pm \varphi(x) \dots\dots$$

have a finite value. This takes place, as is well known, when twice the sum of the algebraic values of the function for characteristic arguments between $x = a$ and $x = b$ is finite, and at the same time the arguments α_1, α_2, $\alpha_3 \dots\dots$ are looked upon as characteristic, but for twice the algebraic sum

$$f(\alpha_1),\ f(\alpha_2),\ f(\alpha_3) \dots\dots$$

the sum of the moduls $f(\alpha_1-\varepsilon)$, $f(\alpha_1+\varepsilon)$, $f(\alpha_2-\varepsilon) \dots$ is substituted.

If now the above mentioned sum be finite, then is

$$\lim^{k=\infty} \left\{ \int_{a}^{\alpha_1-\varepsilon} \varphi(x) \sin kx\, dx + \int_{\alpha_1+\varepsilon}^{\alpha_2-\varepsilon} \varphi(x) \sin kx\, dx \dots + \int_{\alpha_\nu+\varepsilon}^{b} \varphi(x) \sin kx\, dx \right\} = 0$$

$\lim\limits^{k=\infty} \int_{a}^{b} \varphi(x) \sin kx\, dx$ is therefore equal to the sum of the other integrals

$$= \sum_{\nu=1}^{\nu=n} \lim^{k=\infty} \int_{\alpha_\nu-\varepsilon}^{\alpha_\nu+\varepsilon} \varphi(x) \sin kx\, dx.$$

We discover for these integrals a more simple form by the following easy method. We write:

$$\varphi(x) = \frac{F(x)}{x-\alpha_\nu};$$

then is
$$\int_{\alpha_\nu-\varepsilon}^{\alpha_\nu+\varepsilon} \varphi(x) \sin kx\, dx = \int_{\alpha_\nu-\varepsilon}^{\alpha_\nu+\varepsilon} F(x)\, \frac{\sin kx}{x-\alpha_\nu}\, dx.$$

For the more accurate calculation of this integral we can set out the value of any factor of $F(x)$, or even $F(x)$ itself for an argument contained between $\alpha_\nu - \varepsilon$ and $\alpha_\nu + \varepsilon$; for instance let

$$F(x) = f_{,}(x) \cdot f_{,,}(x)$$

and let ξ be the argument beetween $\alpha_\nu - \varepsilon$ and $\alpha_\nu + \varepsilon$ of the mean value of $f_{,}(x)$ then is:

$$\int\limits_{\alpha_\nu-\varepsilon}^{\alpha_\nu+\varepsilon} F(x)\,\frac{\sin kx}{x-\alpha_\nu}\,dx = f_{,}(\xi)\int\limits_{\alpha_\nu-\varepsilon}^{\alpha_\nu+\varepsilon} f_{,,}(x)\,\frac{\sin kx}{x-a_\nu}\,dx.$$

If now we extend the limits $\alpha_\nu-\varepsilon$ and $\alpha_\nu+\varepsilon$, we do not change the value of the integral, if care be only taken that no argument appear within the extended limits for which $f_{,,}(x)$ becomes infinite; since by the extending of the limits according to the previous paragraph we only added a quantity whose value was 0. For example if $f_{,,}(x)$ be such a function that between $-r$ and $+r$ it does not become infinite, then is:

$$\lim^{k=\infty}\int\limits_{\alpha_\nu-\varepsilon}^{\alpha_\nu+\varepsilon} F(x)\,\frac{\sin kx}{x-\alpha_\nu}\,dx = f(\xi)\lim^{k=\infty}\int\limits_{-r}^{+r} f_{,,}(x)\,\frac{\sin kx}{x-\alpha_\nu}\,dx.$$

If we make the positive value ε converge to 0, which is allowed, we do not change the value of the first integral; but the value ξ will approach the value α_ν. Making use of the usual notation to distinguish the two values of a function for one argument for which the function is discontinual, we have

$$\lim^{k=\infty}\int\limits_{\alpha_\nu-\varepsilon}^{\alpha_\nu+\varepsilon} F(x)\,\frac{\sin kx}{x-\alpha_\nu}\,dx = \lim^{k=\infty}\int\limits_{\alpha_\nu-\varepsilon}^{\alpha_\nu} F(x)\,\frac{\sin kx}{x-\alpha_\nu}\,dx + \lim^{k=\infty}\int\limits_{\alpha_\nu}^{\alpha_\nu+\varepsilon} F(x)\,\frac{\sin kx}{x-\alpha_\nu}\,dx$$

$$= f_{,}(\alpha_\nu-0)\lim^{k=\infty}\int\limits_{\alpha_\nu-\varepsilon}^{\alpha_\nu} f_{,,}(x)\,\frac{\sin kx}{x-\alpha_\nu}\,dx + f_{,}(\alpha_\nu+0)\lim^{k=\infty}\int\limits_{\alpha_\nu}^{\alpha_\nu+\varepsilon} f_{,,}(x)\,\frac{\sin kx}{x-\alpha_\nu}\,dx.$$

To repeat briefly the whole method we have adopted. In case, in the integral which we have considered in § 4., a function $\varphi(x)$ appears which for certain arguments contained within its limits e. g. $x=\alpha_\nu$ becomes infinite, we decompose the integral in a series of partial integrals, one half of which contain within their limits no roots of the equation $\varphi(x)=\infty$; these integrals will disappear according to § 3. In the remaining integrals we can make the limits approach the arguments which are roots of the equation: $\varphi(x)=\infty$, so that the limits of each integral differ from each other only by an arbitrarily small finite value. We can now by the method of exhaustion set out those factors of the function which remain finite within the limits of the integral, and afterwards extend the limits of the integral as far as the functions which have been left under the integral admit; if we have e. g. set out all the factors of the function $\varphi(x)$ except the one $\frac{1}{x-\alpha_\nu}$ which contains the root, we can then give to the integral the limits $-\infty$ to $+\infty$.

§ 6. Let $f(x)$ be such a function that it has for every argument x which lies between $x=g$ and $x=h$ a certain finite value. To find the sum of the series

$$\begin{aligned} s = p_0 &+ p_1 e^{-2\pi x\sqrt{-1}} + p_2 e^{-4\pi x\sqrt{-1}} + p_3 e^{-6\pi x\sqrt{-1}} \ldots \text{in inf.} \\ &+ p_{-1} e^{2\pi x\sqrt{-1}} + p_{-2} e^{4\pi x\sqrt{-1}} + p_{-3} e^{6\pi x\sqrt{-1}} \ldots \text{in inf.} \end{aligned}$$

where

$$p_{\pm m} = \int_g^h f(\alpha)\, e^{\pm 2m\pi\alpha\sqrt{-1}}\, d\alpha.$$

that is

$$s = \sum_{n=-\infty}^{n=\infty} p_m e^{-2n\pi x\sqrt{-1}} = \sum_{n=-\infty}^{n=\infty} \int_g^h f(\alpha)\, e^{2n\pi(\alpha-x)\sqrt{-1}}\, d\alpha.$$

The geometrical progression:

$$\begin{aligned} 1 &+ e^{2\pi(\alpha-x)\sqrt{-1}} + e^{4\pi(\alpha-x)\sqrt{-1}} \ldots\ldots + e^{2n\pi(\alpha-x)\sqrt{-1}} \\ &+ e^{-2\pi(\alpha-x)\sqrt{-1}} + e^{-4\pi(\alpha-x)\sqrt{-1}} \ldots\ldots + e^{-n\nu(\alpha-x)\sqrt{-1}} \\ &= \frac{\sin(2n+1)(\alpha-x)\pi}{\sin(\alpha-x)\pi}. \end{aligned}$$

Therefore is

$$s = \lim_{n=\infty} \int_g^h \frac{f(\alpha)\sin(2n+1)(\alpha-x)\pi}{\sin(\alpha-x)\pi}\, d\alpha.$$

Substituting $\alpha - x = \beta$, the limits change into $g - x$ and $h - x$ and

$$s = \lim_{n=\infty} \int_{g-x}^{h-x} \frac{f(x+\beta)\sin(2n+1)\pi\beta}{\sin\pi\beta}\, d\beta.$$

We calculate this value of x by the help of the previous chapter, for the sum of s is an integral of the form:

$$\lim_{k=\infty} \int_a^b \varphi(x)\sin kx\, dx.$$

The fact that in s the increasing n remains a whole number alters nothing in our conclusion. In this case $\varphi(x)$ of § 5. i. e. the factor of $\sin(2n+1)\pi\beta$ has become $\frac{f(x+\beta)}{\sin\pi\beta}$.

Since $f(\beta)$ within the limits $\beta=g$ and $\beta=h$ has a certain and finite value, therefore is $f(x+\beta)$ finite between the arguments $\beta=g-x$ and $\beta=h-x$. It is different however with $\frac{1}{\sin\pi\beta}$ which becomes infinite for every whole β. Assuming that between $g-x$ and $h-x$ the following whole numbers are contained,

$$p,\ p+1, \ldots\ldots q-1,\ q$$

but that $g-x$ and $h-x$ are not whole numbers, then is

$$s=\lim^{n=\infty}\int_{p-\varepsilon}^{p+\varepsilon}\frac{f(x+\beta)\sin(2n+1)\pi\beta}{\sin\pi\beta}d\beta+\lim^{n=\infty}\int_{p+1-\varepsilon}^{p+1+\varepsilon}\frac{f(x+\beta)\sin(2n+1)\pi\beta}{\sin\pi\beta}d\beta+\ldots\ldots,$$

under the conditions expressed in § 5. that the sum of twice the numerical values of the characteristic points of the function $\frac{f(x+\beta)}{\sin\pi x\beta}$ between $g-x$ and $h-x$, or the alternating sum of the function between $g-x$ and $h-x$, is finite, where at the same time the numerical values of

$$f(p-\varepsilon),\ f(p+\varepsilon),\ f(p+1-\varepsilon),\ f(p+1+\varepsilon)\ldots\ldots$$

are substituted for

$$f(p)\ f(p+1)\ldots\ldots f(q-1),\ f(q).$$

These conditions, which we shall discuss more fully in the next paragraph, being fulfilled by the function $f(x)$, we obtain

$$s=\sum_{\mu=p}^{\mu=q}\lim^{n=\infty}\int_{\mu-\varepsilon}^{\mu+\varepsilon}\frac{f(x+\beta)\sin(2n+1)\pi\beta}{\sin\pi\beta}d\beta=\sum_{\mu=p}^{\mu=q}\lim^{n=\infty}\int_{\mu-\varepsilon}^{\mu+\varepsilon}\frac{(\beta-\mu)f(x+\beta)}{\sin\pi\beta}\,\frac{\sin(2n+1)\pi\beta}{\beta-\mu}d\beta.$$

The value of $\frac{\beta-\mu}{\sin\pi\beta}$ for $\beta=\mu$ is $\frac{1}{\pi\cos\pi\mu}=\frac{(-1)^{\mu}}{\pi}$; the value of $f(x+\beta)$ for $\beta=\mu$ is $f(x+\mu)$, if the function is continual for this argument; therefore is:

$$s=\sum_{\mu=p}^{\mu=q}\frac{(-1)^{\mu}}{\pi}f(x+\mu)\ \lim^{n=\infty}\int_{\mu-\varepsilon}^{\mu+\varepsilon}\frac{\sin(2\pi+1)\pi\beta}{\beta-\mu}d\beta.$$

Making $\beta-\mu=\gamma$, the limits of the integral pass over into $-\varepsilon$ and $+\varepsilon$. Further

$$\sin\,(2n+1)\,\pi\beta$$

becomes

$$\sin\,(2n+1)\,(\gamma+\mu)\,\pi = (-1)^{\mu}\,\sin\,(2n+1)\,\gamma\pi,$$

and when we have set $(-1)^{\mu}$ outside of the integral we obtain

$$s = \frac{1}{\pi}\sum_{\mu=p}^{\mu=q} f(x+\mu)\;\lim^{n=\infty}\int_{-\varepsilon}^{+\varepsilon}\frac{\sin\,(2n+1)\,\pi\gamma}{\gamma}\,d\gamma$$

$$= \frac{1}{\pi}\lim^{n=\infty}\int_{-\varepsilon}^{\varepsilon}\frac{\sin\,(2n+1)\,\pi\gamma}{\gamma}\,d\gamma\;\sum_{\mu=q}^{\mu=q} f(x+\mu).$$

We have now to ascertain the value of the integral

$$\lim^{n=\infty}\int_{-\varepsilon}^{\varepsilon}\frac{\sin\,(2n+1)\,\pi\gamma}{\gamma}\,d\gamma.$$

Since this value does not depend upon the form of the function $f(x)$, it might be calculated by the aid of a special case; for instance $f(x)=x$; or having regard to the previous paragraphs as follows. No other argument is possible except the argument $\gamma=0$ which is already contained between the limits $-\varepsilon$ and $+\varepsilon$ which gave us $\frac{1}{\gamma}=\infty$ therefore;

$$\lim^{n=\infty}\int_{-\varepsilon}^{\varepsilon}\frac{\sin\,(2n+1)\,\pi\gamma}{\gamma}\,d\gamma = \lim^{n=\infty}\int_{-\infty}^{\infty}\frac{\sin\,(2n+1)\,\pi\gamma}{\gamma}\,d\gamma = \frac{\pi}{2},$$

since as is well known the integral

$$\int_{-\infty}^{\infty}\frac{\sin\,\alpha x}{x}\,dx = \frac{\pi}{2}$$

for every positive value of α. We have thus summated the series and found its value to be

$$\sum_{\mu=p}^{\mu=q} f(x+\mu)$$

when $p,\ p+1, \ldots . q-1,\ q$ are all the whole numbers which are contained between $g-x$ and $h-x$, and $g-x$ and $h-x$ are not whole numbers, and when moreover $f(x)$ becomes discontinual for any one of the arguments

$$x+p,\ x+p+1, \dots\ x+q-1,\ x+q$$

But if the function $f(x)$ become discontinual for the argument $x+\mu$ we have

$$\lim^{n=\infty} \frac{2}{\pi}\int_{\mu-\varepsilon}^{\mu+\varepsilon} \frac{(\beta-\mu)f(x+\beta)}{\sin\pi\beta}\,\frac{\sin(2n+1)\pi\beta}{\beta-\mu}\,d\beta$$

$$=\lim^{n=\infty} \frac{2}{\pi}\int_{\mu-\varepsilon}^{\mu} \frac{(\beta-\mu)f(x+\beta)}{\sin\pi\beta}\,\frac{\sin(2n+1)\pi\beta}{\beta-\mu}\,d\beta + \lim^{n=\infty} \frac{2}{\pi}\int_{\mu}^{\mu+\varepsilon} \frac{(\beta-\mu)f(x+\beta)}{\sin\pi\beta}\,\frac{\sin(2n+1)\pi\beta}{\beta-\mu}\,d\beta$$

$$=f(x+\mu-0)\ \lim^{n=\infty} \frac{2}{\pi}\int_{-\varepsilon}^{0} \frac{\sin(2n+1)\pi\beta}{\beta}\,d\beta + f(x+\mu+0)\ \lim^{n=\infty} \frac{2}{\pi}\int_{0}^{\varepsilon} \frac{\sin(2n+1)\pi\beta}{\beta}\,d\beta$$

$$=f(x+\mu-0)\ \lim^{n=\infty} \frac{2}{\pi}\int_{-\infty}^{0} \frac{\sin(2n+1)\pi\beta}{\beta}\,d\beta + f(x+\mu+0)\ \lim^{n=\infty} \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin(2n+1)\pi\beta}{\beta}\,d\beta$$

$$=\tfrac{1}{2}f(x+\mu-0) + \tfrac{1}{2}f(x+\mu+0).$$

When finally, $g-x$ or $h-x$ are also whole numbers, we should have added $\frac{1}{2}f(g+x)$ and $\frac{1}{2}f(h+x)$ to the sum $\sum\limits_{\mu=p}^{\mu=q} f(x+\mu)$ which it is easy to find by calculation.

§ 7. The series s easily assumes the usual form of Fourier's series if we make $g=\frac{1}{2}$, $h=\frac{1}{2}$, for it is proved according to the previous paragraph that if

$$-\tfrac{1}{2} < \alpha < \tfrac{1}{2},$$

and function $f(x)$ fulfil the known conditions, that

$$s=\sum_{n=-\infty}^{n=\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2n\pi(\alpha-x)\sqrt{-1}} f(\alpha)\,d\alpha = f(x)$$

or

$$s=\tfrac{1}{2}f(x-0)+\tfrac{1}{2}f(x+0)$$

if the function be discontinual for the argument x.

§ 8. Let us now endeavour to ascertain what the properties of those functions are which can be developed in convergent trigonometric series, and we will at the same time extend this investigation so as to include the general question what sums of values (see. § 6.) can be expressed by the said series. All that is thus far known of the subject is to be found at the end of the famous treatise of Professor *Lejeune-Dirichlet.* *)

It will be seen from the preceding paragraph that every function $f(x)$, and every sum of values of the same for arguments which form an arithmetic series, can be developed in a trigonometric series, when the alternating sum of the function

$$\frac{f(x)}{\sin \pi x}$$

is known to be finite within the limits between which the corresponding integral is to be taken: i. e. when the limits are finite between which the value of the alternating sum is contained. We have thus reduced the question: what functions and what sum of values of the same function admit of development in a trigonometric series, to the simpler one, whether the alternating sum of the same function is finite. Although we cannot hope to have discussed the alternating sum in general in such a way that we can give for every alternating sum the narrowest limists ascertainable by the methods of analysis, we shall yet be able by what we know of these limits, to prove the convergence of a great number of series whose convergence has thus far not been demonstrated; and moreover the reduction of the investigation to the simpler question respecting the alternating sums is progress made. We know that the alternating sum of

$$\frac{f(x)}{\sin \pi x}$$

has a finite and ascertainable value when the numerical values for the characteristic arguments form a converging series.

We have gained thus the advantage that we are not obliged to consider the discontinuities of a function but only its characteristic values; and we can discover whole classes of functions which are infinitely often discontinual, and yet allow of development in *Fourier's* series, since the sum of their characteristic values is finite,

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*) Crelle's Journal Vol. IV. p. 168. Les considérations précédentes prouvent d'une manière rigoureuse que, si la fonction $\varphi(x)$ dont toutes les valeurs sont supposées finies et déterminées ne présente qu'un nombre fini de solutions de continuité entre les limites $-\pi$ et $+\pi$ et si en outre elle n'a qu'un nombre déterminé de maxima et minima entre ces même limites la série . . . . est convergente et a une valeur généralement exprimée par $\frac{1}{2}\varphi(x+\varepsilon)+\frac{1}{2}\varphi(x-\varepsilon)$ où ε désigne un nombre infiniment petit. Il nous resterait à considérer les cas où les suppositions que nous avons faites sur le nombre des solutions de continuité et sur celui des valeurs maxima et minima cessent d'avoir lieu. Ces cas singuliers peuvent être ramenés à ceux, que nous venons de considérer.

§ 9. In case then, the alternating sum of a function is finiter even when the function is discontinual between two succeeding characteristic arguments, and whether the number of discontinuities be finite or infinite, the function can be developed provided of course that the integral of the function have an analytic value, which in fact can very well be the case.

The following example offers us a whole class of such functions as allow of development.

Let $\alpha_1 + \alpha_2 + \alpha_3 + \ldots\ldots \alpha_n \ldots\ldots$ in inf. be an infinite series converging towards the finite and determinate value s, and let all its terms be positive; then will the series

$$\frac{\alpha_1}{s} + \frac{\alpha_2}{s} + \frac{\alpha_3}{s} \ldots\ldots$$

converge towards the value 1. Let, moreover,

$$\beta_1 + \beta_2 + \beta_3 \ldots\ldots$$

be another series the terms of which are all positive, and which also converges toward a finite determinate limit, or in any case is of such a nature that

$$\beta_1 > \beta_2 > \beta_3 > \ldots\ldots$$

Then construct that function $y = F(x)$, the value of which

for every argument x between $x = 0$ and $x = \frac{\alpha_1}{s}$ is $y = \beta_1$;

for every argument x between $x = \frac{\alpha_1}{s}$ and $x = \frac{\alpha_2}{s}$ is $y = \beta_2$; etc.

This function is most easily recognised from its geometrical representation, which exhibits a system of an infinite number of lines parallel with the axis of abscissas.

It is clear that these lines for abscissas which approach the value 1 have ordinates which approach the value 0.

We can develop such a function $F(x)$ in its whole extent, that is so far as it has a value, in a trigonometric series although the number of its discontinuities be infinite. The definite integrals also, of which the developed series consists, do not lose their analytical value for this numerous class of functions because the discontinuities are of such a nature that for the whole amplitude of the integral there exist always between two arguments for which the function becomes discontinual, two others between which it is continual.

Suppose

$$\gamma_1 + \gamma_2 + \gamma_3 \cdots\cdots$$

to be a converging series, the terms of which are indifferently positive or negative but of such a nature that if we give positive values to the negative terms, the series still converges; or in other words of such a nature that the sum of its moduli converges towards a determinate limit; then will a function $f(x)$ of such a nature that

$$f(x) = \gamma_1 \qquad \text{from } x = 0 \text{ to } x = \frac{\alpha_1}{s}$$

$$f(x) = \gamma_2 \qquad \text{from } x = \frac{\alpha_1}{s} \text{ to } x = \frac{\alpha_2}{s} \text{ etc.},$$

allow of development in a trigonometric series, since the alternating sum upon which the development in this case depends is a determinate one.

The simplest case finally, the only one which has thus far been considered by mathematicians, in which the function $f(x)$ has been developed in a series which was true only for arguments which lay between two limits separate from one another by 2π, can be easily discussed by the aid of the previous general law. In this case only the alternating sum of the function

$$\frac{f(x)}{\sin \pi x}$$

between limits separate from one another by unity comes into consideration. Between such limits the maxima and minima or characteristic values of this function are easily developed and we have only to investigate whether the series of the moduli converges toward a determinate limit. Not more difficult is the more general case where the integrals of the trigonometric series have limits farther removed from each other, and in which the sum of the function for arguments which form an arithmetic series are expanded into trigonometric series.

For this purpose the preceding paragraph will be sufficient to determine what is the alternating sum of a function

$$\frac{f(x)}{\sin \pi x},$$

and between what limits is to be investigated. We remark moreover, that the moduli of the values of the function for whole numbers of the arguments x can be better calculated if we take no account of the common factor $\pm \frac{1}{\sin \pi \varepsilon}$.

§ 10. In § 7. we proved the theorem, that:

$$\overset{n=\infty}{\lim} \int_{g-x}^{h-x} \frac{f(x+\beta) \sin (2n+1) \pi\beta}{\sin \pi\beta} d\beta = \sum_{\mu=p}^{\mu=q} f(x+\mu);$$

for the especial case that $x = 0$, we derive therefrom:

$$(1) \quad \sum_{\mu=p}^{\mu=q} f(\mu) = \overset{n=\infty}{\lim} \int_{h}^{g} \frac{f(\beta) \sin (2n+1) \pi\beta}{\sin \pi\beta} d\beta \qquad \text{or}$$

$$f(p) + f(p+1) \ldots + f(q) = \int_{g}^{h} f(\beta)\, d\beta + 2 \int_{g}^{h} f(\beta) \cos 2\beta\, d\beta + 2 \int_{g}^{h} f(\beta) \cos 4\beta\, d\beta + \ldots..$$

where p, $p+1 \ldots.. q$ are all whole numbers which are contained between g and h. The series of the integrals is equal to the sum of the values of the function for all whole numbers as arguments which are contained between g and h, to which also the values

$$\tfrac{1}{2} f(g) \quad \text{and} \quad \tfrac{1}{2} f(h)$$

must be added when g and h are themselves whole numbers. This equation (1) is among the most remarkable of the whole analysis, because by one continual function, as the integrals are, discontinual functions such as the sum of values for determinate arguments are, can be expressed.

If we add to or subtract from the limits of the integrals, finite values which are less than unity, the value of the series will not be thereby altered, provided care be taken, that within the limits of the integral no new whole number appear, or a whole number disappear.

By means of the equation (1) the values of entirely arbitrary and discontinual functions $f(\beta)$ for arguments which form an arithmetic series whose difference is a whole number can be summated, if only the oft repeated condition be fulfilled that the alternating sum of the function between g and h be finite. The Theory of Numbers which has to do with the properties of discontinual functions (of whole numbers), can make use in many cases of the equation (1).

§ 11. Among some interesting applications of the preceding theory, we may notice as the most remarkable the expansion of continued fractions.

Let v be a positive rational number, less than unity, which is developed into the continued fraction

$$v = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c \ddots}}}$$

where $a, b, c \ldots.$ are all positive. If we express $\frac{1}{a}$ as a function of the variable v, then will this function be discontinual, as is easy to be seen from what follows.

We have $\frac{1}{a+1} < v < \frac{1}{a}$,

consequently $a < \frac{1}{v}$ and $a + 1 > \frac{1}{v}$,

and therefore $\frac{1}{v} - 1 < a < \frac{1}{v}$;

then is for every v,

when $\frac{1}{2} < v < 1$, $a = 1$; therefore $\frac{1}{a} = 1$

when $\frac{1}{3} < v \leqq \frac{1}{2}$, $a = 2$; therefore $\frac{1}{a} = \frac{1}{2}$

when $\frac{1}{4} < v \leqq \frac{1}{3}$, $a = 3$; therefore $\frac{1}{a} = \frac{1}{3}$ etc.

if then, we construct such a function of v, $f(v)$ that it has for every argument v a value which is equal to that of a for this argument, the function admits of expansion in a trigonometric series, wherefrom we can deduce the value of $\frac{1}{a}$ for every v.

We have namely

$$\frac{1}{a} = \chi(v) = \sum_{n=-\infty}^{n=\infty} e^{2n\pi v\sqrt{-1}} \int_0^1 \chi(\alpha) e^{2n\pi\alpha\sqrt{-1}} \, d\alpha.$$

furthermore:

$$e^{2n\pi v\sqrt{-1}}\int_0^1 \chi(\alpha)\,e^{2n\pi\alpha\sqrt{-1}}\,d\alpha = e^{2n\pi v\sqrt{-1}}\left\{\int_{\frac{1}{2}}^{1} e^{2n\pi\alpha\sqrt{-1}}\,d\alpha + \frac{1}{2}\int_{\frac{1}{3}}^{\frac{1}{2}} e^{2n\pi\alpha\sqrt{-1}}\,d\alpha \right.$$

$$\left. + \frac{1}{3}\int_{\frac{1}{4}}^{\frac{1}{3}} e^{2n\pi\alpha\sqrt{-1}}\,d\alpha + \ldots\ldots \right.$$

$$\frac{1}{a} = \sum_{n=-\infty}^{n=\infty} \frac{e^{2n\pi v\sqrt{-1}}}{2n\pi\sqrt{-1}}\left\{1 - \frac{1}{2}e^{\frac{2n\pi\sqrt{-1}}{2}} - \frac{1}{2.3}e^{\frac{2n\pi\sqrt{-1}}{3}} - \frac{1}{3.4}e^{\frac{2n\pi\sqrt{-1}}{4}} - \ldots\ldots \text{ in inf.}\right.$$

The series is converging because it is a trigonometric series and fulfills the conditions of the previous paragraphs. We can demonstrate its convergence, moreover, from the fact that

$$\int_{\frac{1}{3}}^{\frac{1}{2}} e^{2n\pi\alpha\sqrt{-1}}\,d\alpha < \tfrac{1}{3} - \tfrac{1}{2}, \quad \int_{\frac{1}{4}}^{\frac{1}{3}} e^{2n\pi\alpha\sqrt{-1}}\,d\alpha < \tfrac{1}{4} - \tfrac{1}{3};$$

it converges in fact more rapidly than the third powers of the reciprocals of the natural numbers. We can with equal ease expand the second partial quotient $\frac{1}{b}$ of the continued fraction for every v into a series. Compared with the previous series, this one will have in place of one term an infinite number of terms; the following is the series referred to:

$$\frac{1}{b} = \sum_{n=-\infty}^{n=\infty} \frac{e^{2n\pi v\sqrt{-1}}}{2n\pi v\sqrt{-1}}\left\{\begin{array}{l} -1 + \frac{1}{1.2}e^{\frac{4n\pi\sqrt{-1}}{5}} + \frac{1}{2.3}e^{\frac{6n\pi\sqrt{-1}}{7}} + \frac{1}{3.4}e^{\frac{8n\pi\sqrt{-1}}{9}} + \frac{1}{4.5}e^{\frac{10n\pi\sqrt{-1}}{11}} + \ldots \\ -e^{\frac{2n\pi\sqrt{-1}}{3}} + \frac{1}{1.2}e^{\frac{4n\pi\sqrt{-1}}{7}} + \frac{1}{2.3}e^{\frac{6n\pi\sqrt{-1}}{10}} + \frac{1}{3.4}e^{\frac{8n\pi\sqrt{-1}}{13}} \\ \qquad + \frac{1}{4.5}e^{\frac{10n\pi\sqrt{-1}}{16}} + \ldots. \\ -e^{\frac{2n\pi\sqrt{-1}}{5}} + \frac{1}{1.2}e^{\frac{4n\pi\sqrt{-1}}{9}} + \frac{1}{2.3}e^{\frac{6n\pi\sqrt{-1}}{13}} + \frac{1}{3.4}e^{\frac{8n\pi\sqrt{-1}}{17}} \\ \qquad + \frac{1}{4.5}e^{\frac{10n\pi\sqrt{-1}}{21}} + \ldots. \\ -\text{ in inf.} \end{array}\right.$$

In such a series every partial quotient can be developed, although they be discontinual values and the convergence of each of these series can be proved from our previous propositions.

§ 12. Let $f(1)\ f(2)\ f3) \ldots\ldots f(n)$ be values of the function $f(x)$ for the arguments

$$1,\ 2,\ 3 \ldots\ldots n;$$

then all symmetric functions of these n values can be easily expanded by means of the foregoing formulas, and moreover taking into consideration that if we regard the arbitrary values

$$a,\ b,\ c \ldots\ldots p,$$

as values of such a function, $f(x)$, that:

$$f(1) = a,\ f(2) = b,$$

which function can always be found, then will the equation of § 7. for every symmetric expression of n elements give an expansion. For

$$\lim^{n=\infty} \int_{\frac{1}{2}}^{n+1} [f(x)]^p \frac{\sin 2k\pi x}{\sin \pi x}\, dx = \int_{\frac{1}{2}}^{n+1} f[(x)]^p\, dx + 2\int_{\frac{1}{2}}^{n+1} f[(x)]^p \cos 2x dx + \ldots \text{ in inf.}$$

gives according to the previous explanation the sum of all p^{th} powers i. e.

$$[f(1)]^p + [f(2)]^p + \ldots\ldots [f(n)]^p$$

and since we can derive from the sum of all the p^{th} powers of n elements every symmetrical function of the same n elements, we can therefore represent every symmetrical function by means of n elements of Fourier's series, which can in fact themselves often be summated.

§ 13. Returning to the proposition:

$$\sum_{\mu=p}^{\mu=q} f(\mu) = \int_g^h f(\alpha)\, d\alpha + \int_g^h f(\alpha)\, e^{2\pi\sqrt{-1}\,\alpha}\, d\alpha + \int_g^h f(\alpha)\, e^{4\pi\sqrt{-1}\,\alpha}\, d\alpha + \ldots .$$

If the integral $\displaystyle\int_g^h f(\alpha)\, e^{2\pi\sqrt{-1}\,\alpha}\, d\alpha$

assume a value which for increasing values of g and h approaches a determinate limit that is to say, if g and h change into $-\infty$ and $+\infty$ and if this limit be expressed by the integral

$$\int_{-\infty}^{\infty} f(\alpha)\, e^{2n\pi\sqrt{-1}\,\alpha}\, d\alpha$$

if moreover at the same time the sum,

$$\sum_{p}^{q} f(n),$$

to which, corresponding to the increase of the integral more and more terms have been annexed, has approached a determinate limit, we have as result

$$\sum_{\mu=-\infty}^{\mu=\infty} f(\mu)=\int_{-\infty}^{\infty} f(\alpha)\, d\alpha+\int_{-\infty}^{\infty} f(\alpha) e^{2\pi\sqrt{-1}\alpha}\, d\alpha+\int_{-\infty}^{\infty} f(\alpha) e^{4\pi\sqrt{-1}\alpha}\, d\alpha+\ldots.$$

We know the value of a great number of integrals of the form

$$\int_{-\infty}^{\infty} f(\alpha)\, e^{k\alpha}\, d\alpha.$$

If now the series $\sum_{-\infty}^{\infty} f(n)$ converges towards a determinate limit, all such integrals furnish equations between series of the most remarkable forms. In investigating whether the series

$$\sum_{-\infty}^{\infty} f(n)$$

converges, we must commence the research with the equation

$$\sum_{p}^{q} f(n)=\sum_{n=-\infty}^{n=\infty} \int_{g}^{h} f(\alpha) e^{2nx\sqrt{-1}}\, d\alpha;$$

extend the limits g and h, whereby at the same time the limits p and q extend themselves: and follow precisely the same method as was employed in determining the value of

$$\int_{-\infty}^{\infty} f(\alpha)\, e^{2nx\sqrt{-1}}\, d\alpha.$$

We thus investigate the convergence of the series

$$\sum_{-\infty}^{+\infty} f(n)$$

under the form

$$\lim_{n=\infty} \sum_{n=-a}^{n=a} f(n);$$

that is $\quad f(0)+f(1)+f(-1)+f(2)+f(-2)+\ldots\ldots$

This is an important consideration which must be attended to, else we obtain faulty series.

If we had derived the formula

$$\tfrac{1}{2}f(0)+\sum_{1}^{\infty}f(n)=\int_{0}^{\infty}f(\alpha)\,d\alpha+\int_{0}^{\infty}f(\alpha)\,e^{2\pi\alpha\sqrt{-1}}\,d\alpha+\ldots..$$

$$+\int_{0}^{\infty}f(\alpha)\,e^{-2\pi\alpha\sqrt{-1}}\,d\alpha+\ldots..$$

from the theorem

$$\tfrac{1}{2}f(0)+\sum_{1}^{\infty}f(n)=\int_{0}^{h}f(\alpha)\,d\alpha+\int_{0}^{h}f(\alpha)\,e^{2\pi\alpha\sqrt{-1}}\,d\alpha+\ldots..$$

$$+\int_{0}^{h}f(\alpha)\,e^{-2\pi\alpha\sqrt{-1}}\,d\alpha+\ldots..$$

by allowing h to converge towards infinity, we should have here to consider, in order to investigate in what cases we can allow the use of the equation, whether

$$\int_{0}^{h}f(\alpha)e^{n\alpha\sqrt{-1}}\,d\alpha$$

with increasing h approaches a limit; and moreover, whether the series

$$\sum_{1}^{q}f(n)$$

with increasing q approaches a limit, i. e. converges. Here should also the arrangement of terms be the same as the extension of the limits of the integrals requires, i. e. the series must be investigated in the order

$$f(1)+f(2)+f(3)\ldots\ldots$$

Let us add a few examples:

In case where $f(\alpha)=\frac{1}{p^2+\alpha^2}$, while p remains constant, we know that

$$\int_{0}^{\infty}f(\alpha)\cos b\alpha\,d\alpha=\frac{\pi}{2p}e^{-pb}$$

now since

$$\tfrac{1}{2}f(0)+\sum_{1}^{\infty}f(\mu)=\int_{0}^{\infty}f(\alpha)\,d\alpha+2\int_{0}^{\infty}f(\alpha)\cos 2\pi\alpha\,d\alpha+2\int_{0}^{\infty}f(\alpha)\cos 4\pi\alpha\,d\alpha+\ldots.,$$

it follows, provided $\sum_{1}^{\infty}f(\mu)$ converges, (which remains to be considered) that

$$\frac{1}{2p^2}+\sum_{1}^{\infty}\frac{1}{p^2+\mu^2}=\frac{\pi}{2p}\left(1+2e^{-2\pi p}+2e^{-4\pi p}+2e^{-6\pi p}+\ldots.\right.$$

But since the series $\sum_{1}^{\infty}\frac{1}{p^2+\mu^2}$ converges, the correct equation ensues:

$$\frac{1}{2p^2}+\frac{1}{p^2+1}+\frac{1}{p^2+4}+\frac{1}{p^2+9}+\ldots\ldots.. \text{ in inf. }=\frac{\pi}{p}\left(\frac{2n^{-2\pi p}-1}{1-e^{-2\pi p}}\right).$$

The value of the integral

$$\int_{-\infty}^{\infty}e^{-(a+b\sqrt{-1})z^2-\sqrt{-1}cz+d}\,dz=\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{c^2}{4(a+b\sqrt{-1})}+d},$$

which is true only for positive values of a, is well known. Supposing

$$a+b\sqrt{-1}=\alpha,\ c=2n\pi,\ d=0$$

this integral gives us:

$$\tfrac{1}{2}+e^{-\alpha}+e^{-4\alpha}+e^{-9\alpha}\ \ldots\ldots.=\sqrt{\frac{\pi}{\alpha}}\left\{\tfrac{1}{2}+e^{-\frac{\pi^2}{\alpha}}+e^{-\frac{4\pi^2}{\alpha}}+\ldots\ldots..\right.$$

Now in case a, which is the real part of the quantity $\alpha=a+b\sqrt{-1}$, be positive, which was in fact our hypothesis, each series will converge; the function also satisfied the condition which was to be fulfilled namely that

$$\sum_{-\infty}^{\infty}\pm\frac{e^{-\alpha z^2}}{\sin\pi z}$$

should become a finite value; it is proved therefore that the equation is correct. In another case

where $\alpha=a+b\sqrt{-1}$,
and in another where $\alpha=a-b\sqrt{-1}$,
we shall have these two equations:

$$(r)\qquad \tfrac{1}{2}+\sum_{n=1}^{n=\infty} e^{-n^2a-n^2b\sqrt{-1}}=\sqrt{\frac{\pi(\alpha-b\sqrt{-1})}{a^2+b^2}}\left\{\tfrac{1}{2}+\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{a^2+b^2}(a-b\sqrt{-1})}\right\}$$

$$(s)\qquad \tfrac{1}{2}+\sum_{n=1}^{n=\infty} e^{-n^2a+nb^2\sqrt{-1}}=\sqrt{\frac{\pi(a+b\sqrt{-1})}{a^2+b^2}}\left\{\tfrac{1}{2}+\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{a^2+b^2}(a+b\sqrt{-1})}\right\}$$

Supposing $\sqrt{\frac{\pi}{a^2+b^2}\,a+\frac{\pi}{a^2+b^2}\,b\sqrt{-1}} \;=\; p+q\sqrt{-1}$,

then is $\sqrt{\frac{\pi}{a^2+b^2}\,a-\frac{\pi}{a^2+b^2}\,b\sqrt{-1}} \;=\; p-q\sqrt{-1}$,

$$p=\sqrt{\pi}\;\cos\tfrac{1}{2}\left(\operatorname{arctg}=\frac{b}{a}\right),\qquad q=\sqrt{\pi}\;\sin\tfrac{1}{2}\left(\operatorname{arctg}=\frac{b}{a}\right),$$

then summating the series (r) and (s), we shall have:

$$1+2\sum_{n=1}^{n=\infty} e^{-n^2a}\cos n^2b=p\left(1+2\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{a^2+b^2}a}\cos\frac{n^2\pi^2}{a^2+b^2}\,b\right)$$
$$+q\left(2\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{a^2+b^2}a}\sin\frac{n^2\pi^2}{a^2+b^2}\,b\right).$$

If we write $b=\frac{\pi}{2}$ we obtain $\cos n^2b=0$, when $n=2m+1$; $\cos n^2b=1$ when $n=2m$; p becomes p', q becomes q'. Therefore is

$$1+2\sum_{n=1}^{n=\infty} e^{-4n^2\alpha}=p'\left(1+2\sum_{n=1}^{n=\infty} e^{-\frac{4n^2\pi^2}{4a^2+\pi^2}\alpha}\cos n^2\,\frac{2\pi^3}{4a^2+\pi^2}\right)$$
$$+q'\left(2\sum_{n=1}^{n=\infty} e^{-\frac{4n^2\pi^2}{4a^2+\pi^2}\alpha}\sin n^2\,\frac{2\pi^3}{4a^2+\pi^2}\right).$$

Supposing $4\alpha=a$, $\alpha=\frac{a}{4}$ we obtain:

$$1+2\sum_{n=1}^{n=\infty} e^{-n^2a}=p'\left(1+2\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{\frac{a^2}{4}+\pi}a}\cos n^2\,\frac{2\pi^3}{\frac{a^2}{4}+\pi^2}\right)$$
$$+q'\left(2\sum_{n=1}^{n=\infty} e^{-\frac{n^2\pi^2}{\frac{a^2}{4}+\pi^2}a}\sin n^2\,\frac{2\pi^3}{\frac{a^2}{4}+\pi^2}\right)$$

consequently a second transformation of the first series gives us, for real and positive values of a:

$$
\begin{aligned}
&1+e^{-a}+e^{-4a}+e^{-9a}+\ldots\ldots \\
&=\tfrac{1}{2}+\sqrt{\frac{\pi}{a}}\left(\tfrac{1}{2}+e^{-\frac{\pi^2}{a}}+e^{-\frac{4\pi^2}{a}}+e^{-\frac{9\pi^2}{a}}+\ldots\ldots\right. \\
&=\tfrac{1}{2}+\sqrt{\frac{\pi}{2}}\,\cos\tfrac{1}{2}\left(\operatorname{arctg}=\frac{\pi}{2a}\right)\left\{\tfrac{1}{2}+e^{-\frac{4\pi^2}{a^2+4\pi^2}a}\cos\frac{8\pi^3}{a^2+4\pi^2}\right. \\
&\qquad\qquad\qquad\qquad\left.+e^{-\frac{16\pi^2}{a^2+4\pi^2}a}\cos\frac{32\pi^3}{a^2+4\pi^2}+\ldots\ldots\right. \\
&\quad+\sqrt{\frac{\pi}{2}}\,\sin\tfrac{1}{2}\left(\operatorname{arctg}=\frac{\pi}{2a}\right)\left\{e^{-\frac{4\pi^2}{a^2+4\pi^2}a}\sin\frac{8\pi^3}{a^2+4\pi^2}\right. \\
&\qquad\qquad\qquad\qquad\left.+e^{-\frac{16\pi^2}{a^2+4\pi^2}a}\sin\frac{32\pi^3}{a^2+4\pi^2}+\ldots\ldots\right.
\end{aligned}
$$

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